Exercise 1F

No need to complete this exercise

1 Simplify:

a $\frac{1}{\sqrt{5}}$ **b** $\frac{1}{\sqrt{11}}$ **c** $\frac{1}{\sqrt{2}}$ **d** $\frac{\sqrt{3}}{\sqrt{15}}$

e $\frac{\sqrt{12}}{\sqrt{48}}$ **f** $\frac{\sqrt{5}}{\sqrt{80}}$ **g** $\frac{\sqrt{12}}{\sqrt{156}}$ **h** $\frac{\sqrt{7}}{\sqrt{63}}$

2 Rationalise the denominators and simplify:

a $\frac{1}{1+\sqrt{3}}$ **b** $\frac{1}{2+\sqrt{5}}$ **c** $\frac{1}{3-\sqrt{7}}$ **d** $\frac{4}{3-\sqrt{5}}$ **e** $\frac{1}{\sqrt{5}-\sqrt{3}}$

f $\frac{3-\sqrt{2}}{4-\sqrt{5}}$ **g** $\frac{5}{2+\sqrt{5}}$ **h** $\frac{5\sqrt{2}}{\sqrt{8}-\sqrt{7}}$ **i** $\frac{11}{3+\sqrt{11}}$ **j** $\frac{\sqrt{3}-\sqrt{7}}{\sqrt{3}+\sqrt{7}}$

k $\frac{\sqrt{17}-\sqrt{11}}{\sqrt{17}+\sqrt{11}}$ **l** $\frac{\sqrt{41}+\sqrt{29}}{\sqrt{41}-\sqrt{29}}$ **m** $\frac{\sqrt{2}-\sqrt{3}}{\sqrt{3}-\sqrt{2}}$

3 Rationalise the denominators and simplify:

a $\frac{1}{(3-\sqrt{2})^2}$ **b** $\frac{1}{(2+\sqrt{5})^2}$ **c** $\frac{4}{(3-\sqrt{2})^2}$

d $\frac{3}{(5+\sqrt{2})^2}$ **e** $\frac{1}{(5+\sqrt{2})(3-\sqrt{2})}$ **f** $\frac{2}{(5-\sqrt{3})(2+\sqrt{3})}$

E/P **4** Simplify $\frac{3-2\sqrt{5}}{\sqrt{5}-1}$ giving your answer in the form $p + q\sqrt{5}$, where p and q are rational numbers. **(4 marks)**

Problem-solving

You can check that your answer is in the correct form by writing down the values of p and q and checking that they are rational numbers.

Mixed exercise 1

1 Simplify:

a $y^3 \times y^5$ **b** $3x^2 \times 2x^5$ **c** $(4x^2)^3 \div 2x^5$ **d** $4b^2 \times 3b^3 \times b^4$

2 Expand and simplify if possible:

a $(x + 3)(x - 5)$ **b** $(2x - 7)(3x + 1)$ **c** $(2x + 5)(3x - y + 2)$

3 Expand and simplify if possible:

a $x(x + 4)(x - 1)$ **b** $(x + 2)(x - 3)(x + 7)$ **c** $(2x + 3)(x - 2)(3x - 1)$

4 Expand the brackets:

a $3(5y + 4)$ **b** $5x^2(3 - 5x + 2x^2)$ **c** $5x(2x + 3) - 2x(1 - 3x)$ **d** $3x^2(1 + 3x) - 2x(3x - 2)$

5 Factorise these expressions completely:

a $3x^2 + 4x$ b $4y^2 + 10y$ c $x^2 + xy + xy^2$ d $8xy^2 + 10x^2y$

6 Factorise:

a $x^2 + 3x + 2$ b $3x^2 + 6x$ c $x^2 - 2x - 35$ d $2x^2 - x - 3$

e $5x^2 - 13x - 6$ f $6 - 5x - x^2$

7 Factorise:

a $2x^3 + 6x$ b $x^3 - 36x$ c $2x^3 + 7x^2 - 15x$

8 Simplify:

a $9x^3 \div 3x^{-3}$ b $(4^{\frac{3}{2}})^{\frac{1}{3}}$ c $3x^{-2} \times 2x^4$ d $3x^{\frac{1}{3}} \div 6x^{\frac{2}{3}}$

9 Evaluate:

a $\left(\frac{8}{27}\right)^{\frac{2}{3}}$ b $\left(\frac{225}{289}\right)^{\frac{3}{2}}$

10 Simplify:

a $\frac{3}{\sqrt{63}}$ b $\sqrt{20} + 2\sqrt{45} - \sqrt{80}$

11 a Find the value of $35x^2 + 2x - 48$ when $x = 25$.

b By factorising the expression, show that your answer to part **a** can be written as the product of two prime factors.

12 Expand and simplify if possible:

a $\sqrt{2}(3 + \sqrt{5})$ b $(2 - \sqrt{5})(5 + \sqrt{3})$ c $(6 - \sqrt{2})(4 - \sqrt{7})$

13 Rationalise the denominator and simplify:

a $\frac{1}{\sqrt{3}}$ b $\frac{1}{\sqrt{2} - 1}$ c $\frac{3}{\sqrt{3} - 2}$ d $\frac{\sqrt{23} - \sqrt{37}}{\sqrt{23} + \sqrt{37}}$ e $\frac{1}{(2 + \sqrt{3})^2}$ f $\frac{1}{(4 - \sqrt{7})^2}$

14 a Given that $x^3 - x^2 - 17x - 15 = (x + 3)(x^2 + bx + c)$, where b and c are constants, work out the values of b and c.

b Hence, fully factorise $x^3 - x^2 - 17x - 15$.

E 15 Given that $y = \frac{1}{64}x^3$ express each of the following in the form kx^n, where k and n are constants.

a $y^{\frac{1}{3}}$ **(1 mark)**

b $4y^{-1}$ **(1 mark)**

E/P 16 Show that $\frac{5}{\sqrt{75} - \sqrt{50}}$ can be written in the form $\sqrt{a} + \sqrt{b}$, where a and b are integers. **(5 marks)**

E 17 Expand and simplify $(\sqrt{11} - 5)(5 - \sqrt{11})$. **(2 marks)**

E 18 Factorise completely $x - 64x^3$. **(3 marks)**

E/P 19 Express 27^{2x+1} in the form 3^y, stating y in terms of x. **(2 marks)**

E/P **20** Solve the equation $8 + x\sqrt{12} = \frac{8x}{\sqrt{3}}$

Give your answer in the form $a\sqrt{b}$ where a and b are integers. **(4 marks)**

P **21** A rectangle has a length of $(1 + \sqrt{3})$ cm and area of $\sqrt{12}$ cm^2.
Calculate the width of the rectangle in cm.
Express your answer in the form $a + b\sqrt{3}$, where a and b are integers to be found. **(2 marks)**

E **22** Show that $\frac{(2-\sqrt{x})^2}{\sqrt{x}}$ can be written as $4x^{-\frac{1}{2}} - 4 + x^{\frac{1}{2}}$. **(2 marks)**

E/P **23** **a** Given that $243\sqrt{3} = 3^a$, find the value of a. **(2 marks)**

b Given further that $3^x \times 27^y = 243\sqrt{3}$, express y as a function of x. **(2 marks)**

E/P **24** Given that $\frac{4x^3 + x^{\frac{5}{2}}}{\sqrt{x}}$ can be written in the form $4x^a + x^b$, write down the value of a and the value of b. **(2 marks)**

Challenge

a Simplify $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$.

b Hence show that $\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \ldots + \frac{1}{\sqrt{24}+\sqrt{25}} = 4$

Summary of key points

1 You can use the laws of indices to simplify powers of the **same base**.
- $a^m \times a^n = a^{m+n}$
- $a^m \div a^n = a^{m-n}$
- $(a^m)^n = a^{mn}$
- $(ab)^n = a^n b^n$

2 Factorising is the opposite of expanding brackets.

3 A quadratic expression has the form $ax^2 + bx + c$ where a, b and c are real numbers and $a \neq 0$.

4 $x^2 - y^2 = (x + y)(x - y)$

5 You can use the laws of indices with any rational power.
- $a^{\frac{1}{m}} = \sqrt[m]{a}$
- $a^{\frac{n}{m}} = \sqrt[m]{a^n}$
- $a^{-m} = \frac{1}{a^m}$
- $a^0 = 1$

6 You can manipulate surds using these rules:
- $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$
- $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

7 The rules to rationalise denominators are:
- Fractions in the form $\frac{1}{\sqrt{a}}$, multiply the numerator and denominator by $\sqrt{a}$.
- Fractions in the form $\frac{1}{a+\sqrt{b}}$, multiply the numerator and denominator by $a - \sqrt{b}$.
- Fractions in the form $\frac{1}{a-\sqrt{b}}$, multiply the numerator and denominator by $a + \sqrt{b}$.

The total revenue, £r, can be calculated by multiplying the number of tickets sold by the price of each ticket. This can be written as $r = p(M - 1000p)$.

b Rearrange r into the form $A - B(p - C)^2$, where A, B and C are constants to be found. **(3 marks)**

c Using your answer to part **b** or otherwise, work out how much the football club should charge for each ticket if they want to make the maximum amount of money. **(2 marks)**

Challenge

Accident investigators are studying the stopping distance of a particular car.

When the car is travelling at 20 mph, its stopping distance is 6 feet.

When the car is travelling at 30 mph, its stopping distance is 14 feet.

When the car is travelling at 40 mph, its stopping distance is 24 feet.

The investigators suggest that the stopping distance in feet, d, is a quadratic function of the speed in miles per hour, s.

a Given that $d(s) = as^2 + bs + c$, find the values of the constants a, b and c.

b At an accident scene a car has left behind a skid that is 20 feet long.
Use your model to calculate the speed that this car was going at before the accident.

Hint Start by setting up three simultaneous equations. Combine two different pairs of equations to eliminate c. Use the results to find the values of a and b first.

No need to do this question

Mixed exercise 2

1 Solve the following equations without a calculator. Leave your answers in surd form where necessary.

a $y^2 + 3y + 2 = 0$ **b** $3x^2 + 13x - 10 = 0$ **c** $5x^2 - 10x = 4x + 3$ **d** $(2x - 5)^2 = 7$

2 Sketch graphs of the following equations:

a $y = x^2 + 5x + 4$ **b** $y = 2x^2 + x - 3$ **c** $y = 6 - 10x - 4x^2$ **d** $y = 15x - 2x^2$

(E) **3** $f(x) = x^2 + 3x - 5$ and $g(x) = 4x + k$, where k is a constant.

a Given that $f(3) = g(3)$, find the value of k. **(3 marks)**

b Find the values of x for which $f(x) = g(x)$. **(3 marks)**

4 Solve the following equations, giving your answers correct to 3 significant figures:

a $k^2 + 11k - 1 = 0$ **b** $2t^2 - 5t + 1 = 0$ **c** $10 - x - x^2 = 7$ **d** $(3x - 1)^2 = 3 - x^2$

5 Write each of these expressions in the form $p(x + q)^2 + r$, where p, q and r are constants to be found:

a $x^2 + 12x - 9$ **b** $5x^2 - 40x + 13$ **c** $8x - 2x^2$ **d** $3x^2 - (x + 1)^2$

(E) **6** Find the value k for which the equation $5x^2 - 2x + k = 0$ has exactly one solution. **(2 marks)**

(E) **7** Given that for all values of x:

$$3x^2 + 12x + 5 = p(x + q)^2 + r$$

a find the values of p, q and r. **(3 marks)**

b Hence solve the equation $3x^2 + 12x + 5 = 0$. **(2 marks)**

(E/P) **8** The function f is defined as $f(x) = 2^{2x} - 20(2^x) + 64,\ x \in \mathbb{R}$.

a Write $f(x)$ in the form $(2^x - a)(2^x - b)$, where a and b are real constants. **(2 marks)**

b Hence find the two roots of $f(x)$. **(2 marks)**

9 Find, as surds, the roots of the equation:

$$2(x + 1)(x - 4) - (x - 2)^2 = 0.$$

10 Use algebra to solve $(x - 1)(x + 2) = 18$.

(E/P) **11** A diver launches herself off a springboard. The height of the diver, in metres, above the pool t seconds after launch can be modelled by the following function:

$$h(t) = 5t - 10t^2 + 10,\ t \geqslant 0$$

a How high is the springboard above the water? **(1 mark)**

b Use the model to find the time at which the diver hits the water. **(3 marks)**

c Rearrange $h(t)$ into the form $A - B(t - C)^2$ and give the values of the constants A, B and C. **(3 marks)**

d Using your answer to part **c** or otherwise, find the maximum height of the diver, and the time at which this maximum height is reached. **(2 marks)**

(E/P) **12** For this question, $f(x) = 4kx^2 + (4k + 2)x + 1$, where k is a real constant.

a Find the discriminant of $f(x)$ in terms of k. **(3 marks)**

b By simplifying your answer to part **a** or otherwise, prove that $f(x)$ has two distinct real roots for all non-zero values of k. **(2 marks)**

c Explain why $f(x)$ cannot have two distinct real roots when $k = 0$. **(1 mark)**

(E/P) **13** Using algebra and showing each stage of your working, find all real solutions to the equation

a $2x + \sqrt{x} - 6 = 0$

b $x^8 - 17x^4 + 16 = 0$ **(3 marks)**

(E/P) **14** Lynn is selling cushions as part of an enterprise project. On her first attempt, she sold 80 cushions at the cost of £15 each. She hopes to sell more cushions next time. Her adviser suggests that she can expect to sell 10 more cushions for every £1 that she lowers the price.

a The number of cushions sold c can be modelled by the equation $c = 230 - Hp$, where £p is the price of each cushion and H is a constant. Determine the value of H. **(1 mark)**

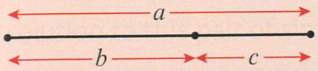
To model her total revenue, £r, Lynn multiplies the number of cushions sold by the price of each cushion. She writes this as $r = p(230 - Hp)$.

b Rearrange r into the form $A - B(p - C)^2$, where A, B and C are constants to be found. **(3 marks)**

c Using your answer to part **b** or otherwise, show that Lynn can increase her revenue by £122.50 through lowering her prices, and state the optimum selling price of a cushion. **(2 marks)**

Challenge

a The ratio of the lengths $a:b$ in this line is the same as the ratio of the lengths $b:c$.



Show that this ratio is $\frac{1+\sqrt{5}}{2}:1$.

b Show also that the infinite square root

$$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}}=\frac{1+\sqrt{5}}{2}$$

Summary of key points

1 To solve a quadratic equation by factorising:

- Write the equation in the form $ax^2 + bx + c = 0$
- Factorise the left-hand side
- Set each factor equal to zero and solve to find the value(s) of x

2 The solutions of the equation $ax^2 + bx + c = 0$ where $a \neq 0$ are given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3 $x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$

4 $ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$

5 The set of possible inputs for a function is called the **domain**.
The set of possible outputs of a function is called the **range**.

6 The **roots** of a function are the values of x for which $f(x) = 0$.

7 You can find the coordinates of a **turning point** of a quadratic graph by completing the square. If $f(x) = a(x + p)^2 + q$, the graph of $y = f(x)$ has a turning point at $(-p, q)$.

8 For the quadratic function $f(x) = ax^2 + bx + c = 0$, the expression $b^2 - 4ac$ is called the **discriminant**. The value of the discriminant shows how many roots $f(x)$ has:

- If $b^2 - 4ac > 0$ then a quadratic function has two distinct real roots.
- If $b^2 - 4ac = 0$ then a quadratic function has one repeated real root.
- If $b^2 - 4ac < 0$ then a quadratic function has no real roots.

9 Quadratics can be used to model real-life situations.

Mixed exercise 3

(E) **1** $2kx - y = 4$

$4kx + 3y = -2$

are two simultaneous equations, where k is a constant.

a Show that $y = -2$. **(3 marks)**

b Find an expression for x in terms of the constant k. **(1 mark)**

(E) **2** Solve the simultaneous equations:

$x + 2y = 3$

$x^2 - 4y^2 = -33$ **(7 marks)**

(E) **3** Given the simultaneous equations

$x - 2y = 1$

$3xy - y^2 = 8$

a Show that $5y^2 + 3y - 8 = 0$. **(2 marks)**

b Hence find the pairs (x, y) for which the simultaneous equations are satisfied. **(5 marks)**

(E) **4** **a** By eliminating y from the equations

$x + y = 2$

$x^2 + xy - y^2 = -1$

show that $x^2 - 6x + 3 = 0$. **(2 marks)**

b Hence, or otherwise solve the simultaneous equations

$x + y = 2$

$x^2 + xy - y^2 = -1$

giving x and y in the form $a \pm b\sqrt{6}$, where a and b are integers. **(5 marks)**

(E) **5** **a** Given that $3^x = 9^{y-1}$, show that $x = 2y - 2$. **(1 mark)**

b Solve the simultaneous equations:

$x = 2y - 2$

$x^2 = y^2 + 7$ **(6 marks)**

(E) **6** Solve the simultaneous equations:

$x + 2y = 3$

$x^2 - 2y + 4y^2 = 18$ **(7 marks)**

(E/P) **7** The curve and the line given by the equations

$kx^2 - xy + (k + 1)x = 1$

$-\frac{k}{2}x + y = 1$

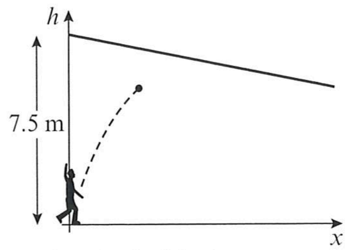
where k is a non-zero constant, intersect at a single point.

a Find the value of k. **(5 marks)**

b Give the coordinates of the point of intersection of the line and the curve. **(3 marks)**

E/P **8** A person throws a ball in a sports hall. The height of the ball, h m, can be modelled in relation to the horizontal distance from the point it was thrown from by the quadratic equation:

$$h = -\frac{3}{10}x^2 + \frac{5}{2}x + \frac{3}{2}$$



The hall has a ceiling which slopes downwards in a straight line from an initial height of 7.5 m at the point the ball is thrown. The height of the ceiling reduces by 20 cm for every metre of horizontal distance from the point the ball is thrown.

Determine whether the model predicts that the ball will hit the ceiling. **(5 marks)**

E **9** Give your answers in set notation.

a Solve the inequality $3x - 8 > x + 13$. **(2 marks)**

b Solve the inequality $x^2 - 5x - 14 > 0$. **(4 marks)**

E **10** Find the set of values of x for which $(x - 1)(x - 4) < 2(x - 4)$. **(6 marks)**

E **11** **a** Use algebra to solve $(x - 1)(x + 2) = 18$. **(2 marks)**

b Hence, or otherwise, find the set of values of x for which $(x - 1)(x + 2) > 18$. Give your answer in set notation. **(2 marks)**

12 Find the set of values of x for which:

a $6x - 7 < 2x + 3$ **(2 marks)**

b $2x^2 - 11x + 5 < 0$ **(4 marks)**

c $5 < \frac{20}{x}$ **(4 marks)**

d both $6x - 7 < 2x + 3$ and $2x^2 - 11x + 5 < 0$. **(2 marks)**

E **13** Find the set of values of x that satisfy $\frac{8}{x^2} + 1 \leqslant \frac{9}{x}$, $x \neq 0$ **(5 marks)**

E **14** Find the values of k for which $kx^2 + 8x + 5 = 0$ has real roots. **(3 marks)**

E/P **15** The equation $2x^2 + 4kx - 5k = 0$, where k is a constant, has no real roots.

Prove that k satisfies the inequality $-\frac{5}{2} < k < 0$. **(3 marks)**

E **16** **a** Sketch the graphs of $y = \text{f}(x) = x^2 + 2x - 15$ and $\text{g}(x) = 6 - 2x$ on the same axes. **(4 marks)**

b Find the coordinates of any points of intersection. **(3 marks)**

c Write down the set of values of x for which $\text{f}(x) > \text{g}(x)$. **(1 mark)**

E **17** Find the set of values of x for which the curve with equation $y = 2x^2 + 3x - 15$ is below the line with equation $y = 8 + 2x$. **(5 marks)**

E **18** On a coordinate grid, shade the region that satisfies the inequalities:

$y > x^2 + 4x - 12$ and $y < 4 - x^2$. **(5 marks)**

E/P **19** **a** On a coordinate grid, shade the region that satisfies the inequalities

$y + x < 6$, $y < 2x + 9$, $y > 3$ and $x > 0$. **(6 marks)**

b Work out the area of the shaded region. **(2 marks)**

Challenge

1 Find the possible values of k for the quadratic equation $2kx^2 + 5kx + 5k - 3 = 0$ to have real roots.

2 A straight line has equation $y = 2x - k$ and a parabola has equation $y = 3x^2 + 2kx + 5$ where k is a constant. Find the range of values of k for which the line and the parabola do not intersect.

Summary of key points

1 Linear simultaneous equations can be solved using elimination or substitution.

2 Simultaneous equations with one linear and one quadratic equation can have up to two pairs of solutions. You need to make sure the solutions are paired correctly.

3 The solutions of a pair of simultaneous equations represent the points of intersection of their graphs.

4 For a pair of simultaneous equations that produce a quadratic equation of the form $ax^2 + bx + c = 0$:
- $b^2 - 4ac > 0$ two real solutions
- $b^2 - 4ac = 0$ one real solution
- $b^2 - 4ac < 0$ no real solutions

5 The solution of an inequality is the set of all real numbers x that make the inequality true.

6 To solve a quadratic inequality:
- Rearrange so that the right-hand side of the inequality is 0
- Solve the corresponding quadratic equation to find the critical values
- Sketch the graph of the quadratic function
- Use your sketch to find the required set of values.

7 The values of x for which the curve $y = f(x)$ is **below** the curve $y = g(x)$ satisfy the inequality $f(x) < g(x)$.
The values of x for which the curve $y = f(x)$ is **above** the curve $y = g(x)$ satisfy the inequality $f(x) > g(x)$.

8 $y < f(x)$ represents the points on the coordinate grid below the curve $y = f(x)$.
$y > f(x)$ represents the points on the coordinate grid above the curve $y = f(x)$.

9 If $y > f(x)$ or $y < f(x)$ then the curve $y = f(x)$ is not included in the region and is represented by a dotted line.
If $y \geqslant f(x)$ or $y \leqslant f(x)$ then the curve $y = f(x)$ is included in the region and is represented by a solid line.